More is Universal:
an Introduction to the Conformal Field Theory

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Outline

Lecture I

1. Motivation: Criticality and Universality
2. Conformal Transformations
3. Defining Conformal Field Theory
4. The Operator Product Expansion (OPE) and Conformal Bootstrap

Next Lecture (?)

5. Radial Quantization and the State–Operator Correspondence
6. Two-Dimensional CFT

Section 1 of 6

Motivation: Criticality and Universality

The ϕ^4 theory: a familiar interacting model

A standard continuum description of a scalar order parameter is

$$S[\phi] = \int \mathrm{d}^d x \left[\underbrace{\frac{1}{2}(\partial_\mu \phi)^2}_{\text{kinetic}} + \underbrace{\frac{r}{2}\phi^2}_{\text{mass (tuning parameter)}} + \underbrace{\frac{u}{4!}\phi^4}_{\text{interaction}} - \underbrace{h\phi}_{\text{external field}} \right].$$

At $h = 0$, the action is invariant under

$$\mathbb{Z}_2 : \qquad \phi(x) \longmapsto -\phi(x), \qquad S[\phi] = S[-\phi].$$

The problem

What can we learn about this interacting theory?

The classical strategy finds a stationary field

The principle of stationary action gives

$$\frac{\delta S}{\delta \phi(x)} = 0 \quad \Longrightarrow \quad -\partial^2\phi + r\phi + \frac{u}{3!}\phi^3 - h = 0.$$

For a spatially uniform saddle, $\partial_\mu \phi = 0$, this reduces to

$$r\phi + \frac{u}{3!}\phi^3 - h = 0.$$

What it provides

- Candidate equilibrium configurations
- The mean-field phase structure

What it misses

- Fluctuations around the saddle
- Physics of continuous phase transition (when $r \to 0$)

Scale invariance as the Emergent Symmetry

Describe the local fluctuation by

$$\delta\phi(x) = \phi(x) - \langle\phi\rangle\,, \qquad G(x) = \langle\delta\phi(0)\,\delta\phi(x)\rangle = \langle\phi(0)\phi(x)\rangle_c\,.$$

Away from criticality, a finite correlation length ξ permits an exponential decay, schematically $G(x) \sim \mathrm{e}^{-|x|/\xi}$. At the phase transition $r \to 0$, however,

$$\xi \longrightarrow \infty.$$

Renormalization group argument demonstrates that scale invariance is present at the critical point.
Concretely, under the rescale of the theory $\tilde{x} = \lambda x$,

$$[\delta\tilde{\phi}(\tilde{x})] = \lambda^{-\Delta_\phi}[\delta\phi(x)]$$

where Δ_ϕ is the scaling dimension of ϕ. Thus,

$$G(\lambda x) = \lambda^{-2\Delta_\phi} G(x)$$

The Wilson–Fisher fixed point reveals universality

The RG reorganizes the problem:

$$\text{microscopic theory} \xrightarrow{\text{coarse grain and rescale}} \text{RG flow} \xrightarrow{\text{long distances}} \text{fixed point.}$$

In $d = 4 - \epsilon$, the dimensionless quartic coupling has the schematic flow

$$\beta(g) = -\epsilon g + A g^2 + \mathcal{O}(g^3), \qquad A > 0,$$

with a non-Gaussian fixed point

$$g_* = \frac{\epsilon}{A} + \mathcal{O}(\epsilon^2).$$

The Wilson–Fisher lesson

Different microscopic systems can flow to the same fixed point and share the same scaling dimensions and critical exponents: universality.

Symmetry already solves part of the theory

Suppose $h = 0$ and the state preserves the $\mathbb{Z}_2$ symmetry. Then

$$\langle \phi \rangle = \langle -\phi \rangle = - \langle \phi \rangle \quad \Longrightarrow \quad \boxed{\langle \phi \rangle = 0}.$$

More generally, every odd correlation function vanishes:

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(x_{2n+1}) \rangle = 0.$$

The important point

We obtained exact information about observables without explicitly solving the partition function.

The conclusion assumes a $\mathbb{Z}_2$-invariant state; a chosen symmetry-broken vacuum may have $\langle \phi \rangle \neq 0$.

More symmetry means fewer unknowns

Symmetry	What it tells us
$\mathbb{Z}_2$	Selection rules: odd correlators vanish
Translations and rotations	Correlators depend only on relative geometry
Scale invariance	Correlators acquire power-law behavior
Conformal invariance	Two- and three-point functions are strongly constrained; four-point functions obey consistency conditions

The question that leads to CFT

Can an enlarged symmetry determine the critical theory without solving the underlying field theory?

This is the conformal-field-theory strategy.

Section 2 of 6

Conformal Transformations

Conformal maps preserve angles, not distances

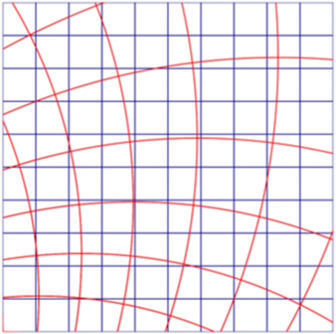
A conformal transformation may distort lengths and areas while preserving the angles at which curves intersect.

In the figure:

- the blue grid is mapped to the red grid;
- the spacing between neighboring lines changes;
- the intersection angles remain unchanged.

Geometric idea

A conformal map acts locally like a rotation followed by a position-dependent rescaling.



A two-dimensional conformal deformation.

A conformal map rescales the metric locally

A map $\mathcal{M} : x^\mu \mapsto \tilde{x}^\mu(x)$ is conformal when

$$\boxed{\tilde{g}_{\rho\sigma}(\tilde{x})\frac{\partial \tilde{x}^\rho}{\partial x^\mu}\frac{\partial \tilde{x}^\sigma}{\partial x^\nu} = \Omega^2(x) g_{\mu\nu}(x)}$$

for some positive local scale factor $\Omega(x)$.
Equivalently, the line element transforms as

$$\mathrm{d}\tilde{s}^2 = \Omega^2(x)\mathrm{d}s^2.$$

Since every inner product is multiplied by the same factor,

$$\frac{\tilde{g}(\tilde{u},\tilde{v})}{\sqrt{\tilde{g}(\tilde{u},\tilde{u})}\sqrt{\tilde{g}(\tilde{v},\tilde{v})}} = \frac{g(u,v)}{\sqrt{g(u,u)}\sqrt{g(v,v)}}.$$

Thus the angle between u and v is unchanged.

Infinitesimal conformal maps obey one equation

In flat space, take an infinitesimal transformation

$$\tilde{x}^\mu = x^\mu + \epsilon^\mu(x), \qquad \Omega^2(x) = 1 + 2\sigma(x).$$

Expanding the metric condition to first order gives

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\sigma(x)\eta_{\mu\nu}.$$

Taking the trace determines the local rescaling:

$$\sigma(x) = \frac{1}{d}\partial_\rho \epsilon^\rho.$$

Therefore the allowed vector fields satisfy the conformal Killing equation

$$\boxed{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}}.$$

Solving the equation leaves four possibilities

Differentiating the conformal Killing equation and permuting indices gives

$$\partial_\mu \partial_\nu \epsilon_\rho = \eta_{\mu\rho} \partial_\nu \sigma + \eta_{\nu\rho} \partial_\mu \sigma - \eta_{\mu\nu} \partial_\rho \sigma.$$

For $d \geq 3$, the integrability conditions make $\sigma(x)$ affine, so $\epsilon^\mu(x)$ is at most quadratic:

$$\boxed{\epsilon^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + 2(b \cdot x) x^\mu - b^\mu x^2}, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}.$$

a^μ	translation
$\omega_{\mu\nu}$	rotation
λ	dilatation, or scale transformation
b^μ	special conformal transformation

In $d = 2$, these generate the global conformal subgroup; local conformal transformations form an infinite-dimensional symmetry.

The four essential generators

The four parts of $\epsilon^\mu(x)$ correspond to the differential operators

$$
\begin{aligned}
P_\mu &= -\mathrm{i}\partial_\mu, && \text{translation,} \\
L_{\mu\nu} &= \mathrm{i}\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right), && \text{rotation,} \\
D &= -\mathrm{i}x^\mu\partial_\mu, && \text{dilatation,} \\
K_\mu &= -\mathrm{i}\left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right), && \text{special conformal transformation.}
\end{aligned}
$$

The result of the calculation

Solving the metric condition has reduced every global infinitesimal conformal transformation to P_μ, $L_{\mu\nu}$, D, and K_μ.

Three finite transformations are immediate

Exponentiating P_μ, $L_{\mu\nu}$, and D gives

Transformation	**Finite map**	$\boldsymbol{\Omega^2}$
Translation	$\tilde{x}^\mu = x^\mu + a^\mu$	1
Rotation	$\tilde{x}^\mu = R^\mu{}_\nu x^\nu, \quad R^\mathsf{T} R = \mathbf{1}$	1
Dilatation	$\tilde{x}^\mu = \lambda x^\mu, \quad \lambda > 0$	λ^2

- Translations and rotations are isometries: they preserve distances.
- A dilatation rescales every distance by the same constant λ.
- A special conformal transformation instead has a position-dependent scale factor.

A special conformal transformation uses inversion

Begin with inversion,

$$I: \qquad x^\mu \longmapsto \tilde{x}^\mu = \frac{x^\mu}{x^2}.$$

A special conformal transformation is the composition

$$I \circ T_{-b} \circ I.$$

Its finite form is

$$\boxed{\tilde{x}^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}}.$$

The corresponding local rescaling is

$$\mathrm{d}\tilde{s}^2 = \frac{\mathrm{d}s^2}{\left(1 - 2b \cdot x + b^2 x^2\right)^2} \Rightarrow \Omega(x) = \frac{1}{1 - b \cdot x + b^2 x^2}.$$

Transition to conformal field theory

How do fields and correlation functions transform if the theory is conformally invariant?

Section 3 of 6

Defining Conformal Field Theory

Primary fields follow the local scale factor

Let $\mathcal{O}_a(x)$ be a primary operator of scaling dimension Δ. Under a conformal transformation,

$$x^\mu \mapsto \tilde{x}^\mu, \qquad \mathrm{d}\tilde{s}^2 = \Omega^2(x)\mathrm{d}s^2,$$

it transforms homogeneously:

$$\tilde{\mathcal{O}}_a(\tilde{x}) = \Omega(x)^{-\Delta} D[R(x)]_a{}^b \mathcal{O}_b(x).$$

The matrix $D[R(x)]$ accounts for spin. For the rest of this section, we focus on a scalar primary $\phi(x)$, for which

$$\boxed{\tilde{\phi}(\tilde{x}) = \Omega(x)^{-\Delta}\phi(x)}.$$

Meaning of Δ

The scaling dimension measures how the local operator responds to a local change of length scale.

The four transformations of a scalar primary

Define

$$\sigma_b(x) = 1 - 2b \cdot x + b^2 x^2, \qquad \Omega(x) = \sigma_b(x)^{-1}$$

for a special conformal transformation.

Transformation	**Coordinates**	**Scalar primary**
Translation	$\tilde{x}^\mu = x^\mu + a^\mu$	$\tilde{\phi}(\tilde{x}) = \phi(x)$
Rotation	$\tilde{x}^\mu = R^\mu{}_\nu x^\nu$	$\tilde{\phi}(\tilde{x}) = \phi(x)$
Dilatation	$\tilde{x}^\mu = \lambda x^\mu$	$\tilde{\phi}(\tilde{x}) = \lambda^{-\Delta}\phi(x)$
Special conformal	$\tilde{x}^\mu = \dfrac{x^\mu - b^\mu x^2}{\sigma_b(x)}$	$\tilde{\phi}(\tilde{x}) = \sigma_b(x)^\Delta \phi(x)$

A spinning operator would also acquire the appropriate rotation matrix $D[R(x)]$.

Each operator insertion contributes one scale factor

Define the n-point function $G_n(x_1, \ldots, x_n) = \langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle$.
Conformal covariance gives

$$\boxed{\tilde{G}_n(\tilde{x}_1, \ldots, \tilde{x}_n) = \prod_{i=1}^{n} \Omega(x_i)^{-\Delta_i} G_n(x_1, \ldots, x_n)}.$$

In particular,

$$G_n(\lambda x_1, \ldots, \lambda x_n) = \lambda^{-\sum_i \Delta_i} G_n(x_1, \ldots, x_n),$$

while an SCT gives

$$G_n(\tilde{x}_1, \ldots, \tilde{x}_n) = \prod_{i=1}^{n} \sigma_b(x_i)^{\Delta_i} G_n(x_1, \ldots, x_n).$$

Why this becomes a constraint

The transformed and original correlators describe the same physical function.

Three symmetries almost fix the two-point function

Consider two scalar primaries with dimensions Δ_1 and Δ_2:

$$G_{12}(x_1, x_2) = \langle \phi_1(x_1)\phi_2(x_2) \rangle \,, \qquad x_{12} = x_1 - x_2.$$

Applying the symmetries one at a time,

$$\begin{aligned} G_{12}(x_1, x_2) &= F(x_{12}), && \text{translation,} \\ F(x_{12}) &= f(|x_{12}|), && \text{rotation,} \\ f(\lambda|x_{12}|) &= \lambda^{-(\Delta_1+\Delta_2)} f(|x_{12}|), && \text{dilatation.} \end{aligned}$$

The only possible power law is therefore

$$\boxed{G_{12}(x_1, x_2) = \frac{C_{12}}{|x_{12}|^{\Delta_1+\Delta_2}}}.$$

Special conformal symmetry completes the answer

Under an SCT, pairwise distances transform as

$$|\tilde{x}_{12}|^2 = \frac{|x_{12}|^2}{\sigma_b(x_1)\sigma_b(x_2)}.$$

The candidate power law produces equal powers of $\sigma_b(x_1)$ and $\sigma_b(x_2)$, whereas covariance requires the powers Δ_1 and Δ_2 separately. Consistency therefore requires

$$\Delta_1 = \Delta_2 \qquad \text{or} \qquad C_{12} = 0.$$

Hence

$$\boxed{\langle \phi_i(x_1)\phi_j(x_2)\rangle = \begin{cases} \dfrac{C_{ij}}{|x_{12}|^{2\Delta}}, & \Delta_i = \Delta_j = \Delta, \\ 0, & \Delta_i \neq \Delta_j. \end{cases}}$$

Within operators of equal dimension, the two-point matrix can be diagonalized and normalized to $C_{ij} = \delta_{ij}$.

Three points introduce three distances

Translation and rotation invariance imply

$$G_{123} = \langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = G(|x_{12}|, |x_{23}|, |x_{31}|).$$

Use the power-law ansatz

$$G_{123} = \frac{C_{123}}{|x_{12}|^{\alpha}|x_{23}|^{\beta}|x_{31}|^{\gamma}}.$$

Scale covariance gives only one condition:

$$\alpha + \beta + \gamma = \Delta_1 + \Delta_2 + \Delta_3.$$

Scale invariance is not enough

One equation cannot determine the three exponents α, β, and γ separately.

Special conformal symmetry fixes all three exponents

Using

$$|\tilde{x}_{ij}|^2 = \frac{|x_{ij}|^2}{\sigma_b(x_i)\sigma_b(x_j)},$$

match the local scale factor associated with each operator insertion:

$$\alpha + \gamma = 2\Delta_1, \qquad \alpha + \beta = 2\Delta_2, \qquad \beta + \gamma = 2\Delta_3.$$

Therefore,

$$\alpha = \Delta_1 + \Delta_2 - \Delta_3, \quad \beta = \Delta_2 + \Delta_3 - \Delta_1, \quad \gamma = \Delta_3 + \Delta_1 - \Delta_2.$$

$$\boxed{G_{123} = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}.$$

Conformal symmetry leaves dynamical data to determine

Conformal symmetry determines the coordinate dependence of scalar two- and three-point functions:

$$\langle \phi_i(x_1)\phi_j(x_2)\rangle \sim \frac{\delta_{ij}}{|x_{12}|^{2\Delta_i}},$$

while

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\rangle \sim C_{ijk} \times \text{the conformally fixed position dependence.}$$

- The scaling dimensions Δ_i are theory-dependent.
- The constants C_{ijk} are three-point, or OPE, coefficients.
- Together, $\{\Delta_i, C_{ijk}\}$ form the local dynamical data of a CFT.

The question that leads to the bootstrap

How can we exploit the full set of constraints imposed by conformal invariance to determine the remaining CFT data?

Section 4 of 6

The Operator Product Expansion and Conformal Bootstrap

Two nearby operators can fuse into one

When two local operators approach one another, their product can be represented by a sum of local operators at a single point:

$$\boxed{\mathcal{O}_i(x)\mathcal{O}_j(0) \sim \sum_k f_{ij}{}^k(x)\, \mathcal{O}_k(0)}.$$

The fusion idea

From far away, two nearby insertions cannot be resolved separately. Their combined effect is reproduced by every local operator allowed by the symmetries.

The symbol $\sim$ means an expansion valid as $x \to 0$ inside correlation functions. The operators on the right include primaries and their descendants.

Operator product expansion

The OPE turns a product of local observables into a sum over the local operator spectrum of the theory.

Scaling fixes the distance dependence

Under $x \mapsto \lambda x$, dimensional consistency requires

$$f_{ij}{}^{k}(\lambda x) = \lambda^{\Delta_k - \Delta_i - \Delta_j} f_{ij}{}^{k}(x).$$

For scalar operators, rotational symmetry therefore gives

$$f_{ij}{}^{k}(x) \propto |x|^{\Delta_k - \Delta_i - \Delta_j}.$$

The OPE takes the schematic form

$$\boxed{\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k C_{ijk} |x|^{\Delta_k - \Delta_i - \Delta_j} \left[\mathcal{O}_k(0) + \text{descendants}\right]}.$$

- Conformal symmetry fixes the relative descendant contributions.
- The undetermined information is Δ_k, the spin of $\mathcal{O}_k$, and the OPE coefficient C_{ijk}.

Symmetry selects the allowed fusion channels

Return to the $\mathbb{Z}_2$ symmetry of the Ising universality class:

$$\sigma \mapsto -\sigma, \qquad \epsilon \mapsto \epsilon.$$

Parity restricts the OPE before we calculate anything:

$$\begin{aligned} \sigma \times \sigma &= \mathbf{1} + \sum_{\mathcal{O}^+} \mathcal{O}^+, \\ \sigma \times \mathcal{O}^+ &= \sum_{\mathcal{O}^-} \mathcal{O}^-, \\ \mathcal{O}^+ \times \mathcal{O}^+ &= \mathbf{1} + \sum_{\mathcal{O}^+} \mathcal{O}^+. \end{aligned}$$

In particular,

$$\boxed{\sigma \times \sigma = \mathbf{1} + \epsilon + \epsilon' + T_{\mu\nu} + \cdots}.$$

The term *fusion rule* specifies which symmetry sectors may appear; the OPE need not contain only finitely many operators.

Four points contain the first unfixed function

Conformal symmetry fixes scalar two- and three-point functions, but four points leave a function of two conformal cross-ratios:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{\mathcal{G}(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}}.$$

The cross-ratios are

$$\boxed{u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}}.$$

Why four points matter

The function $\mathcal{G}(u,v)$ contains genuine dynamical information, but the OPE lets us calculate it in more than one way.

The same four points admit different fusions

s-channel

$$(\phi_1\phi_2)(\phi_3\phi_4)$$

$$\phi_1 \times \phi_2 \to \mathcal{O}, \qquad \phi_3 \times \phi_4 \to \mathcal{O}.$$

Fuse the pairs (12) and (34).

t-channel

$$(\phi_1\phi_4)(\phi_2\phi_3)$$

$$\phi_1 \times \phi_4 \to \mathcal{O}, \qquad \phi_2 \times \phi_3 \to \mathcal{O}.$$

Fuse the pairs (14) and (23).

Consistency requirement

These are two expansions of the same four-point function, so their answers must agree for every allowed configuration of the four points.

A conformal block collects one exchanged multiplet

Applying the OPE in the $(12)(34)$ channel gives

$$\boxed{\mathcal{G}(u,v) = \sum_{\mathcal{O}\in\phi\times\phi} C^2_{\phi\phi\mathcal{O}}\, g_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(u,v)}.$$

- $\mathcal{O}$ labels an exchanged primary operator.
- $g_{\Delta,\ell}(u,v)$ is its conformal block.
- One block contains the primary and every descendant in its conformal multiplet.
- Conformal symmetry fixes the block once Δ and ℓ are specified.

Theory-dependent versus universal

The block is universal kinematics; the spectrum and $C_{\phi\phi\mathcal{O}}$ specify the theory.

Crossing symmetry is OPE associativity

Equality of the two fusion channels gives

$$\boxed{\begin{aligned} 0 = \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \Big[& v^{\Delta_\phi} g_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(u,v) \\ & - u^{\Delta_\phi} g_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(v,u) \Big]. \end{aligned}}$$

This is the crossing equation. Conceptually, it is the statement

$$\boxed{(\mathcal{O}_1\mathcal{O}_2)\mathcal{O}_3 = \mathcal{O}_1(\mathcal{O}_2\mathcal{O}_3)}.$$

The bootstrap principle

Different orders of local fusion must reconstruct exactly the same correlation functions.

Unitarity makes crossing restrictive

The unknown CFT data are

$$\{\Delta_{\mathcal{O}}, \ell_{\mathcal{O}}, C_{ijk}\}\,.$$

They must satisfy several simultaneous constraints:

Principle	**Consequence**
Crossing symmetry	different OPE channels agree
Unitarity	$C^2_{\phi\phi\mathcal{O}} \geq 0$
Unitarity bounds	lower bounds on $\Delta_{\mathcal{O}}$ at fixed spin
Internal symmetry	only allowed representations appear
Local CFT structure	identity and stress tensor are present

Numerical bootstrap

Test whether a proposed spectrum can satisfy all these conditions.

The three-dimensional Ising CFT forms an island

The critical Ising CFT contains a lowest $\mathbb{Z}_2$-odd scalar σ and a lowest $\mathbb{Z}_2$-even scalar ϵ:

$$\sigma \times \sigma = \mathbf{1} + \epsilon + \cdots .$$

- Crossing and unitarity for $\langle \sigma\sigma\sigma\sigma \rangle$ produce a sharp corner in the allowed region near the Ising theory.
- Combining σ and ϵ correlators, together with mild spectral-gap assumptions, isolates a small allowed island.

A mixed-correlator bootstrap determination gives

$$\boxed{\Delta_\sigma = 0.5181489(10), \qquad \Delta_\epsilon = 1.412625(10)}$$

and

$$C_{\sigma\sigma\epsilon} = 1.0518537(41).$$

F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, JHEP **08** (2016) 036, arXiv:1603.04436.

More constraints reveal more universal data

The lecture began with an interacting critical theory:

$$\phi^4 \quad \longrightarrow \quad \text{Wilson–Fisher fixed point.}$$

Conformal symmetry reorganizes the problem:

$$\text{equations of motion} \quad \longrightarrow \quad \text{CFT data } \{\Delta_i, C_{ijk}\}.$$

The bootstrap then constrains those data through

$$\boxed{\text{conformal symmetry} + \text{OPE consistency} + \text{unitarity} \Longrightarrow \text{universal critical data}}.$$

Answer to the opening question

More symmetry does not merely simplify the theory: combined with consistency, it can determine sharply constrained properties of a strongly interacting critical point.

End of Lecture I

Next Lecture

Radial Quantization and Two-Dimensional CFT

Section 5 of 6

Radial Quantization and the State–Operator Correspondence

Concentric circles become cylinder slices

Write a point on the punctured plane in polar coordinates:

$$\vec{r} = x\hat{x} + y\hat{y} = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y}, \qquad r > 0.$$

Introduce a logarithmic radial coordinate,

$$\boxed{\rho = \ln r}, \qquad \theta \sim \theta + 2\pi.$$

punctured plane	$\longleftrightarrow$	infinite cylinder
$0 < r < \infty$	$\longleftrightarrow$	$-\infty < \rho < \infty$
circle $r =$ constant	$\longleftrightarrow$	slice $\rho =$ constant

Geometry made manifest

$$\mathbb{R}^2 \setminus \{0\} \cong \mathbb{R}_\rho \times S^1, \qquad r \to 0 \iff \rho \to -\infty.$$

A circle carries a quantum state

Fixing ρ leaves the periodic coordinate

$$\theta \sim \theta + 2\pi,$$

so every constant-ρ slice is a spatial circle S^1.

$$\boxed{|\Psi(\rho)\rangle \in \mathcal{H}_{S^1}}$$

- A path integral over the region inside the circle prepares $|\Psi(\rho)\rangle$ on its boundary.
- A path integral over an annulus evolves a state from a smaller circle to a larger circle.
- The Hilbert space is associated with a one-dimensional spatial manifold; it is not itself one-dimensional.

Choice of Euclidean time

We use ρ as Euclidean time: outward radial evolution means increasing ρ.

Radial evolution is generated by dilatations

Translation in cylinder time is a scale transformation on the plane:

$$\rho \mapsto \rho + \alpha \qquad \Longleftrightarrow \qquad r \mapsto \mathrm{e}^{\alpha} r.$$

Therefore radial evolution is generated by the dilatation operator,

$$|\Psi(\rho_2)\rangle = \mathrm{e}^{-(\rho_2-\rho_1)D}|\Psi(\rho_1)\rangle, \qquad \boxed{H_{\text{cyl}} = D}$$

for a unit-radius cylinder.
If a state is an eigenstate of D,

$$D|\mathcal{O}\rangle = \Delta_{\mathcal{O}}|\mathcal{O}\rangle,$$

then its cylinder energy gap is

$$\boxed{E_{\mathcal{O}} - E_0 = \Delta_{\mathcal{O}}}.$$

The payoff

Scaling dimensions become measurable energy spacings.

An operator at the origin prepares a state

With no insertion, the path integral over a disk prepares the vacuum:

$$\text{empty disk} \qquad \longrightarrow \qquad |0\rangle \in \mathcal{H}_{S^1}.$$

Inserting a local operator inside the disk prepares a new boundary state:

$$\text{disk containing } \mathcal{O}(0) \qquad \longrightarrow \qquad |\mathcal{O}\rangle.$$

Because the origin is the infinite past of the cylinder,

$$\boxed{|\mathcal{O}\rangle = \lim_{r\to 0} \mathcal{O}(r\hat{n})|0\rangle = \mathcal{O}(0)|0\rangle}.$$

State–operator correspondence

Local scaling operators at the origin are in one-to-one correspondence with energy eigenstates on S^1.

Primaries seed towers of cylinder states

A primary operator produces a state satisfying

$$D|\mathcal{O}\rangle = \Delta_{\mathcal{O}}|\mathcal{O}\rangle, \qquad K_\mu|\mathcal{O}\rangle = 0.$$

Acting with translations produces its descendants:

$$|\mathcal{O}\rangle, \qquad P_\mu|\mathcal{O}\rangle, \qquad P_\mu P_\nu|\mathcal{O}\rangle, \qquad \dots$$

Level	**State**	**Scaling dimension**
0	$\|\mathcal{O}\rangle$	$\Delta_{\mathcal{O}}$
1	$P_\mu\|\mathcal{O}\rangle$	$\Delta_{\mathcal{O}} + 1$
n	$P_{\mu_1}\cdots P_{\mu_n}\|\mathcal{O}\rangle$	$\Delta_{\mathcal{O}} + n$

Cylinder interpretation

A conformal multiplet appears as an energy tower built above a primary state.

The critical Ising chain realizes the cylinder

Consider the periodic transverse-field Ising chain at its critical point:

$$\boxed{H_{\text{Ising}} = -\sum_{j=1}^{N} \left(X_j X_{j+1} + Z_j \right), \qquad X_{N+1} = X_1}.$$

The ring of N spins is a lattice approximation to the spatial circle S^1. Its low-energy eigenstates approximate states of the Ising CFT on the cylinder.
For a circle of circumference L,

$$E_\alpha(L) - E_0(L) = \frac{2\pi v}{L} \Delta_\alpha + \text{finite-size corrections},$$

and therefore

$$\boxed{\Delta_\alpha \simeq \frac{L}{2\pi v} \left(E_\alpha - E_0 \right)}.$$

A numerical test of the correspondence

Diagonalizing the critical spin chain reveals the spectrum of local CFT operators.

Low-energy levels reveal the Ising operator spectrum

The Ising CFT contains three primary operators:

Primary	$\mathbb{Z}_2$ **parity**	**Exact** Δ
Identity $\mathbf{1}$	even	0
Spin σ	odd	$\frac{1}{8}$
Energy density ϵ	even	1

Numerical simulation of periodic chains found

$$\Delta_\sigma = 0.1249995, \qquad \Delta_\epsilon = 0.9999994,$$

and identified the leading continuum fields of simple lattice operators:

$$X_j \longleftrightarrow \sigma, \qquad X_j X_{j+1} - Z_j \longleftrightarrow \epsilon,$$

up to normalization and higher-dimension corrections.

More than a spectrum

The same construction works in any dimension

Write a point in $\mathbb{R}^d \setminus \{0\}$ as

$$x^\mu = r n^\mu, \qquad n^\mu n_\mu = 1, \qquad n \in S^{d-1}.$$

Therefore the punctured space is a family of spheres labeled by their radius:

$$\mathbb{R}^d \setminus \{0\} \cong \mathbb{R}^+ \times S^{d-1}.$$

Introducing $\rho = \ln r$ turns a dilatation into a translation:

$$r \mapsto \mathrm{e}^\alpha r \qquad \Longleftrightarrow \qquad \rho \mapsto \rho + \alpha.$$

Thus fixed-ρ slices are spheres S^{d-1} and radial evolution is generated by D.

local operator in $\mathbb{R}^d$	$\longleftrightarrow$	state in $\mathcal{H}_{S^{d-1}}$
scaling dimension	$\longleftrightarrow$	cylinder energy above the vacuum
rotation representation	$\longleftrightarrow$	angular momentum on S^{d-1}

Radial quantization in one sentence

The spectrum of local CFT operators becomes the energy spectrum of a quantum theory living

Section 6 of 6

Two-Dimensional CFT