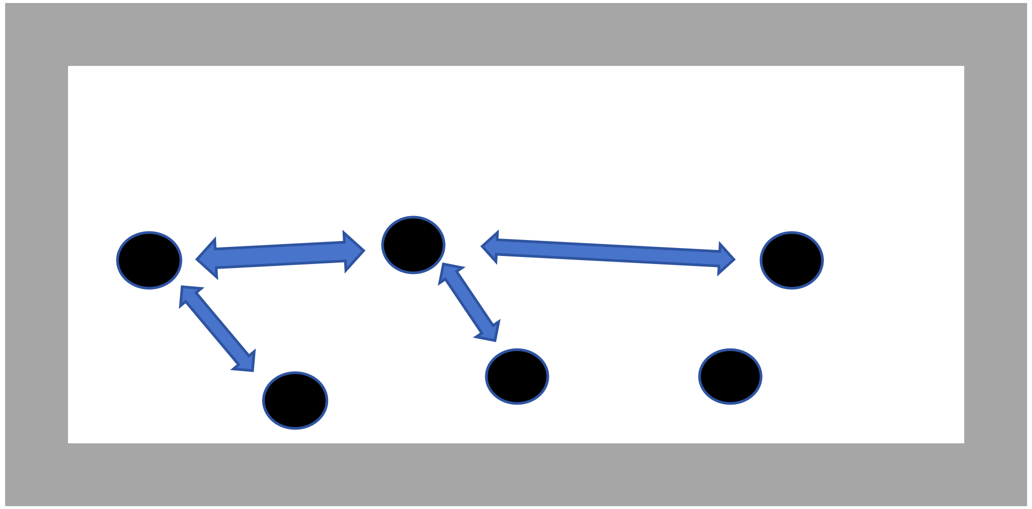
Current Algebra of HK Model

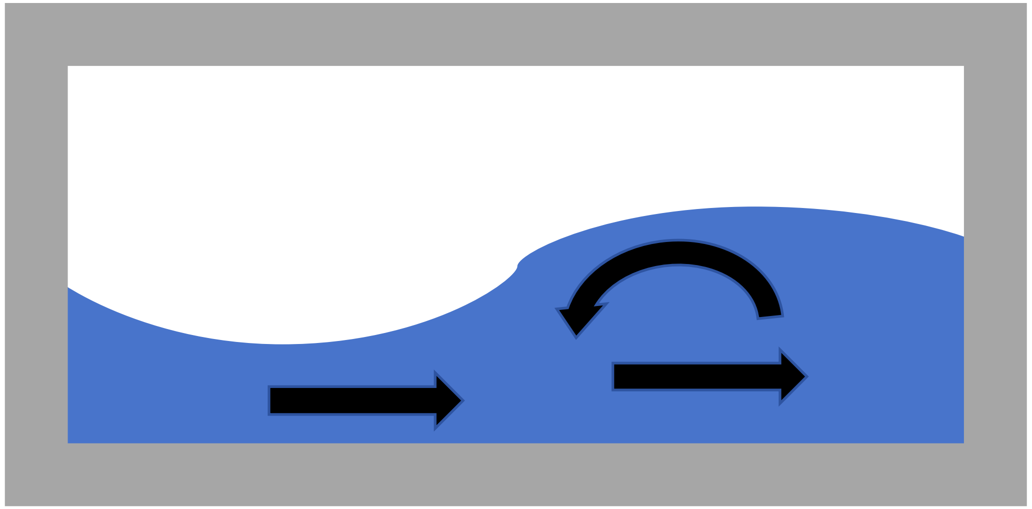
Yuting Bai

(1)Current Algebra Description of Free Fermion



Particle Description

$$c_{\mathbf{k}}, c_{\mathbf{k}}^{\dagger}$$

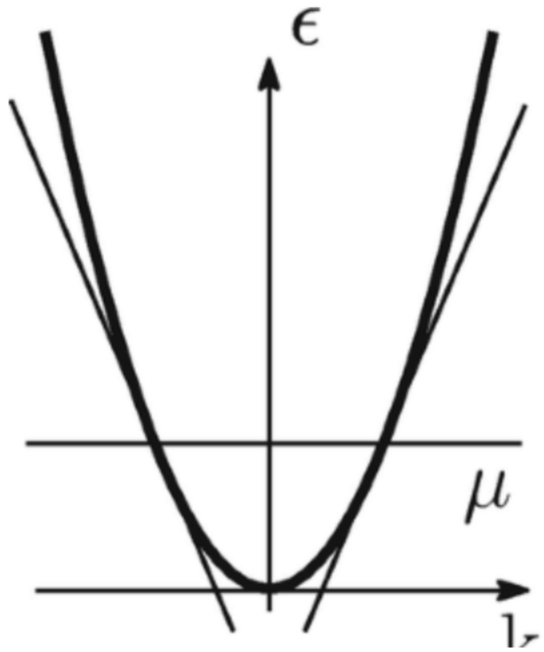


Fluid Description

$$\rho_k(q) = c_{k+q}^{\dagger} c_k$$

(1)Current Algebra of Free Fermion

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} n_{\mathbf{k}} \simeq \sum_{\mathbf{k}} v_F k (c^\dagger_{\mathbf{kR}} c_{\mathbf{kR}} - c^\dagger_{\mathbf{kL}} c_{\mathbf{kL}})$$



For free fermion, we check the operator dynamics in the Heisenberg picture.

Particle

$$c_{\mathbf{kR}}(t) = e^{-iv_F kt} c_{\mathbf{kR}}$$

$$c^\dagger_{\mathbf{kR}}(t) = e^{iv_F kt} c^\dagger_{\mathbf{kR}}$$

Current

$$\rho_{qR}(t) = \sum_{\mathbf{k}} e^{iv_F(k+q)t} c^\dagger_{\mathbf{k+qR}} c_{\mathbf{kR}} e^{-iv_F kt}$$

$$= e^{iv_F qt} \rho_{qR}$$

Both of them acquire a U(1) phase under evolution!

$$\rho_q \equiv \frac{1}{L} \int dx e^{iqx} \rho(x) = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k+q}} c_{\mathbf{k}}$$

(1)Current Algebra of Free Fermion

- Getting the Equation of Motion correct is not enough.
- Does current operators ρ_q obey boson's commutation relation?

$$
\begin{aligned}
&[\rho_{\mathbf{q},\mathbf{a}}, \rho_{\mathbf{q}',\mathbf{b}}] \\
=&\frac{\delta_{\mathbf{ab}}}{N}\sum_{\mathbf{k_1},\mathbf{k_2}}{}' [c^\dagger_{\mathbf{q}+\mathbf{k_1},\mathbf{a}} c_{\mathbf{k_1},\mathbf{b}}, c^\dagger_{\mathbf{q}'+\mathbf{k_2},\mathbf{a}} c_{\mathbf{k_2},\mathbf{b}}] \\
=&\frac{\delta_{\mathbf{ab}}}{N}\sum_{\mathbf{k_1},\mathbf{k_2}}{}' (\delta_{\mathbf{q}'+\mathbf{k_2},\mathbf{k_1}} c^\dagger_{\mathbf{q}+\mathbf{k_1},\mathbf{a}} c_{\mathbf{k_2},\mathbf{b}} - \delta_{\mathbf{q}+\mathbf{k_1},\mathbf{k_2}} c^\dagger_{\mathbf{q}'+\mathbf{k_2},\mathbf{a}} c_{\mathbf{k_1},\mathbf{b}}) \\
=&\frac{\delta_{\mathbf{ab}}}{N}\sum_{\mathbf{k_1}}{}' (c^\dagger_{\mathbf{q}+\mathbf{k_1},\mathbf{a}} c_{-\mathbf{q}'+\mathbf{k_1},\mathbf{b}} - c^\dagger_{\mathbf{q}+\mathbf{q}'+\mathbf{k_1},\mathbf{a}} c_{\mathbf{k_1},\mathbf{b}}) \stackrel{?}{=} 0
\end{aligned}
$$

Do we fail?

(1)Current Algebra of Free Fermion

- The canonical commutation relation does not give you the right answer!

$$\frac{\delta_{\mathbf{ab}}}{N}\sum_{\mathbf{k_1}}{}'(c^{\dagger}_{\mathbf{q+k_1,a}}c_{\mathbf{-q'+k_1,b}}-c^{\dagger}_{\mathbf{q+q'+k_1,a}}c_{\mathbf{k_1,b}})$$

The summation is performed on the interval $[-\Lambda,\Lambda]$

$$\Downarrow$$

$$\frac{\delta_{\mathbf{ab}}}{N}\left(\sum_{\mathbf{k_1}=-\Lambda-\mathbf{q'}}^{-\Lambda}c^{\dagger}_{\mathbf{q+q'+k_1,a}}c_{\mathbf{k_1,b}}-\sum_{\mathbf{k_1}=\Lambda-\mathbf{q'}}^{\Lambda}c^{\dagger}_{\mathbf{q+q'+k_1,a}}c_{\mathbf{k_1,b}}\right)$$

$$\Downarrow$$

$$[\rho_{\mathbf{q,a}},\rho_{\mathbf{q',b}}]=\delta_{\mathbf{ab}}\,\mathrm{sgn}(\mathrm{a})\frac{L\mathbf{q}}{2\pi}\delta_{\mathbf{q+q',0}}$$

The commutator of current operator is a c-number. It is boson!

(2)Bjorken-Johnson-Low Prescription

- The previous method works well for free fermion. Here we introduce a method that calculates the anomalous commutator directly from correlation function.

$$T(\omega) = \int dt e^{i\omega t} \langle a| TA(t)B(0) |b\rangle$$

$$\lim_{\omega\to\infty} \omega T(\omega) = i \langle a| [A(0), B(0)]_- |b\rangle .$$

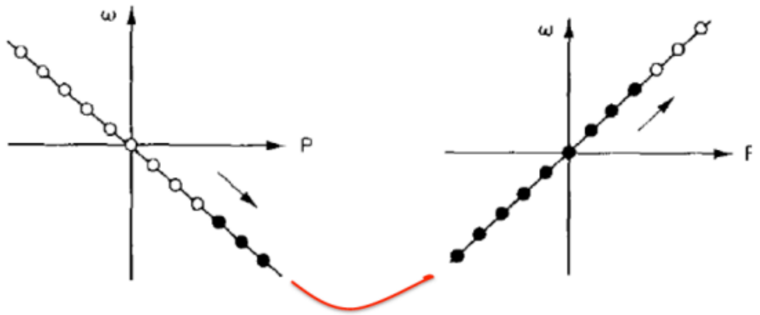
The matrix element of anomalous commutator depends on the high-frequency limit. This implies the robustness of anomalous commutator to interaction!

(2)Bjorken-Johnson-Low Prescription

- One can show that BJL prescription reproduce the U(1) Kac Moody algebra for chiral fermion.

$$
\begin{aligned}
T(\omega) =& \delta_{\mathbf{ab}}\delta_{\mathbf{q}+\mathbf{q}',0}\int dt e^{i\omega t}\sum_{\mathbf{k_1},\mathbf{k_2}}\delta_{\mathbf{k_1}+\mathbf{q},\mathbf{k_2}}e^{-i\mathrm{sgn}(\mathbf{a})v_F q t}f_a(\mathbf{k_1},\mathbf{k_2}) \\
=& i\delta_{\mathbf{ab}}\delta_{\mathbf{q}+\mathbf{q}',0}\sum_{\mathbf{k_1}}\left(\frac{n_{\mathbf{k_1}+\mathbf{q},\mathbf{a}}(1-n_{\mathbf{k_1},\mathbf{a}})}{\omega-\mathrm{sgn}(\mathbf{a})v_F q+i0^+}-\frac{n_{\mathbf{k_1},\mathbf{a}}(1-n_{\mathbf{k_1}+\mathbf{q},\mathbf{a}})}{\omega-\mathrm{sgn}(\mathbf{a})v_F q-i0^+}\right)
\end{aligned}
\tag{20}
$$

$$\lim_{\omega\to\infty}\omega T(\omega)$$



$$
\langle 0|\left[n_{\mathbf{q},\mathbf{a}},n_{\mathbf{q}',\mathbf{b}}\right]|0\rangle = i\delta_{\mathbf{ab}}\delta_{\mathbf{q}+\mathbf{q}',0}\sum_{\mathbf{k_1}}\left[n_{\mathbf{k_1}+\mathbf{q},\mathbf{a}}(1-n_{\mathbf{k_1},\mathbf{a}})-n_{\mathbf{k_1},\mathbf{a}}(1-n_{\mathbf{k_1}+\mathbf{q},\mathbf{a}})\right]
$$

(3)HK Model

- HK Model: A NFL theory that breaks Z_2 symmetry.

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} n_{\mathbf{k}\sigma} + U \sum_{\mathbf{k}} n_{\mathbf{k}\uparrow} n_{\mathbf{k}\downarrow}$$

- The HK model has Green function zero, which leads to violation of Luttinger Theorem.

Could one find a local solvable theory that has the similar correlation function with HK in the low-energy, long wavelength limit?

$$O(4)$$

$$\downarrow$$

$$\cancel{Z_2} \times SO(4)$$

$$N_{tot} \neq V_{FS}$$

(4)Current Algebra for HK

- In HK model, the eigenparticles are not bare fermions, but holon and doublon!

$$c_{k\sigma}(t) = e^{-i\epsilon_k t} c_{k\sigma}(1 - n_{k\bar{\sigma}}) + e^{-i(\epsilon_k + U)t} c_{k\sigma} n_{k\bar{\sigma}}$$

- Instead of bare current, one should work with parton current

$$\rho_{q,a\sigma} = \sum_{\alpha,\beta=\xi,\eta} \rho^{\alpha\beta}_{q,a\sigma}$$

$$\rho^{\alpha\beta}_{q,a\sigma} = \sum_k (c^{\alpha}_{k+q,a\sigma})^\dagger c^{\beta}_{k,a\sigma}$$

$$c^{\xi}_{k\sigma} = c_{k\sigma} P^{\xi}_{k\bar{\sigma}} = c_{k\sigma}(1 - n_{k\bar{\sigma}})$$
$$c^{\eta}_{k\sigma} = c_{k\sigma} P^{\eta}_{k\bar{\sigma}} = c_{k\sigma} n_{k\bar{\sigma}}$$

$\rho^{\xi\xi}$,$\rho^{\eta\eta}$: Projection of current to Upper/Lower Hubbard Band

$\rho^{\xi\eta}$,$\rho^{\eta\xi}$: Mixture of Upper/Lower Hubbard Band

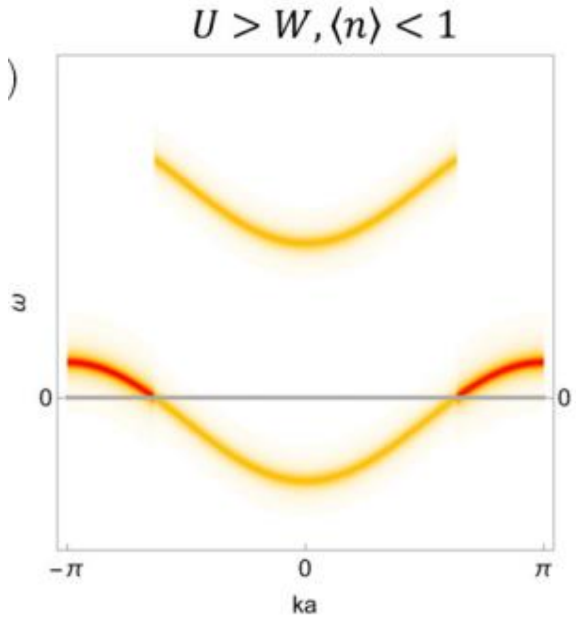
(4) Current Algebra of HK

- We may use BJL prescription to calculate anomalous commutators between these parton currents
- We consider ground state with filling surface at Low Hubbard Band

(1) Since no double occupancy, there is no support for J^{UU}

(2)We assume the perturbation is small enough that one can **approximate the commutator with the ground state average value**.

(3)We average over all degenerate ground state in HK when evaluate commutator using BJL



(4)Current Algebra of HK

- We may start with projected density $\rho^{\xi\xi}$.

$$[\rho^{\xi\xi}_{qa}, \rho^{\xi\xi}_{q'a'}] = -\frac{\delta_{aa'}}{2}\mathrm{sgn}(a)\delta_{q+q',0}\frac{qL}{2\pi}$$

- In general, the projected current $\rho^{\xi\xi}$ is not conserved subjected to an external field.

The Projected Charge

$$N_L = N - \sum_{\mathbf{k}} n_{\mathbf{k}\uparrow} n_{\mathbf{k}\downarrow}$$

Add Onsite Potential

$$H = H_{HK} + \sum_{\mathbf{k}\sigma} V_{-\mathbf{k}\sigma}\rho_{\mathbf{k}\sigma}$$

$$[N_L, H] = -\sum_{\mathbf{k}\sigma} V_{-\mathbf{k}\sigma}(\rho^{\eta\xi}_{\mathbf{k}\sigma} - \rho^{\xi\eta}_{\mathbf{k}\sigma})$$

Must include J^{LU} , J^{UL} to satisfy conservation law!!

(4)Current Algebra of HK

$$\left[\rho^{\xi\xi}_{qa}, \rho^{\xi\xi}_{q'a'}\right] = -\frac{\delta_{aa'}}{2}\mathrm{sgn}(a)\delta_{q+q',0}\frac{qL}{2\pi}$$
$$\left[\rho^{\xi\xi}_{qa}, \rho^{\xi\eta}_{q'a'}\right] = \delta_{aa'}\rho^{\xi\eta}_{q+q'a}$$
$$\left[\rho^{\xi\xi}_{qa}, \rho^{\eta\xi}_{q'a'}\right] = -\delta_{aa'}\rho^{\eta\xi}_{q+q'a}$$
$$\left[\rho^{\eta\xi}_{qa}, \rho^{\xi\eta}_{q'a'}\right] = -\frac{\delta_{aa'}}{2}\left(\rho^{\xi\xi}_{q+q'a} + \frac{1}{2}\mathrm{sgn}(a)\delta_{q+q',0}\frac{qL}{2\pi}\right),$$

To get the commutator $[\rho^{\eta\xi}, \rho^{\xi\eta}]$, there are two ways
(1)using the Jacobi identity.
(2)using BJL.
The latter yields an extra nonlocal equal-time commutator, which is suppressed by $\frac{q}{k_F}$.

The 1st/4th line is derived from BJL prescription.
The 2nd/3rd line is fixed by conservation law.

(4)Current Algebra of HK

The parton current in the charge sector is affine su(2) Lie algebra!

$$\left[\varrho^{i}_{qa\sigma}, \varrho^{j}_{q'a'\sigma'}\right] = \delta_{aa'}\delta_{\sigma\sigma'}\left(i\epsilon_{ijk}\varrho^{k}_{q+q'a} - \frac{\mathrm{sgn}(a)}{2}\delta_{q+q',0}\frac{qL}{2\pi}\right)$$

$$\begin{aligned}\varrho^{z} &= \rho^{\xi\xi}\\ \varrho^{x} &= \frac{1}{2}(\rho^{\xi\eta} + \rho^{\eta\xi})\\ \varrho^{y} &= \frac{1}{2i}(\rho^{\xi\eta} - \rho^{\eta\xi})\end{aligned}$$

(5)Hamiltonian

- Under the Heisenberg picture, the current operator only gain a phase under time evolution.

$$\rho_{qa\sigma}^{\xi\xi}(t) = e^{i\epsilon_{qa}t}\rho_{qa\sigma}^{\xi\xi}$$
$$\rho_{qa\sigma}^{\xi\eta}(t) = e^{i(\epsilon_{qa}-U)t}\rho_{qa\sigma}^{\xi\eta}$$
$$\rho_{qa\sigma}^{\eta\xi}(t) = e^{i(\epsilon_{qa}+U)t}\rho_{qa\sigma}^{\eta\xi}$$

$$[H(\rho_{qa\sigma}^{\alpha\beta}), \rho_{qa\sigma}^{\alpha\beta}] = E_{qa\sigma}^{\alpha\beta}\rho_{qa\sigma}^{\alpha\beta}$$

(5)Hamiltonian

- We can use a LOCAL hamiltonian to generate the correct energy.

$$H = v_F \frac{2\pi}{L} \sum_{i=x,y,z} \sum_{qa\sigma} \varrho^i_{qa\sigma} \varrho^i_{-qa\sigma} - U \sum_{a\sigma} \varrho^z_{q=0a\sigma}.$$

Jz commutes with J^2 if there is no anomaly. Schwinger term gives you kinetic energy vq.

Zeeman field that generates the energy difference U

(5)Hamiltonian

- We can use a LOCAL hamiltonian to generate the correct energy.

$$H = v_F \frac{2\pi}{L} \sum_{i=x,y,z} \sum_{qa\sigma} \varrho^i_{qa\sigma} \varrho^i_{-qa\sigma} - U \sum_{a\sigma} \varrho^z_{q=0a\sigma}.$$

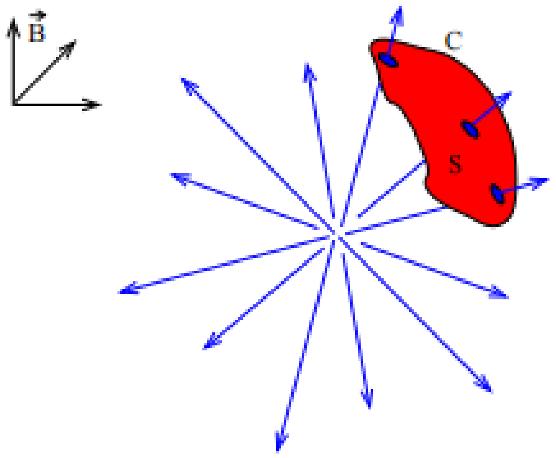
$$\Downarrow$$

$$H = 2\pi v_F \sum_{a\sigma} \int dx \vec{\varrho}^{\,2}_{a\sigma}(x) - \sum_{a\sigma} \int dx U \varrho^z_{a\sigma}(x).$$

(6)Topology and Geometry Aspect

- We could modify the second term.

$$H[\mathbf{U}(t)] = \sum_{a\sigma} \int dx [2\pi v_F \vec{\varrho}^{\,2}_{a\sigma}(x) - \mathbf{U}(x,t) \cdot \vec{\varrho}_{a\sigma}(x)].$$



Berry Phase

$$S_{\text{eff}}[\vec{U}] = c \int dt \int_0^1 ds \vec{U} \cdot (\partial_t \vec{U} \times \partial_s \vec{U}) + \ldots$$

The effective action must
e **multi-valued** function
of U!

Relation with anomaly
in LW Functional?

(6)Correlation function

- We may compare correlation function from our Hamiltonian and the Kubo formalism

Kubo: Imaginary time-ordered correlation function

$$\langle\rho\rho\rangle(q,\omega) = \left(\rho - \frac{|q|}{2\pi}\right)\frac{U}{\omega^2 - U^2} + \frac{v_F q^2}{\pi\omega^2} + o_{q\to 0}(q^2)$$

Current Algebra: Real time retarded correlation function

$$\langle\rho\rho\rangle_B^R(q,\omega) = \frac{\rho U}{\omega^2 - U^2} + \frac{v_F q^2}{\pi\omega^2} + O\left(q^2\right)$$

The difference is controlled by $\frac{q}{k_F}$ and $\frac{v_F q}{U}$.
In the limit $k_F, U \to \infty$, the difference goes to zero.

(7)New Interpretation for HK model

- We show that a local model could restore the equation of motion and two-body correlation of band HK model at IR limit.
- The local, well-defined excitation in HK model are particle-hole pair instead of single particle operators!
- Maybe we could interpret the nonlocality of HK model as a result of rewriting local degree of freedom in nonlocal way.

$$(U_x + iU_y)\rho^{\xi\eta}_{qa\sigma} \Leftrightarrow \sum_k (U_x + iU_y)(c^\xi)^\dagger_{k+qa\sigma} c^\eta_{ka\sigma},$$

Local SU(2) Object

Nonlocal particle-hole cloud

(11)To Do List

- (1)Ambiguity of BJL Prescription
- Replacing commutator with ground state expectation value might miss something

$$
\begin{aligned}
\left[\rho^{\xi\xi}_{qa}, \rho^{\xi\xi}_{q'a'}\right] &= -\frac{\delta_{aa'}}{2}\mathrm{sgn}(a)\delta_{q+q',0}\frac{qL}{2\pi} \\
\left[\rho^{\xi\xi}_{qa}, \rho^{\xi\eta}_{q'a'}\right] &= \boxed{\delta_{aa'}\rho^{\xi\eta}_{q+q'a}} \\
\left[\rho^{\xi\xi}_{qa}, \rho^{\eta\xi}_{q'a'}\right] &= \boxed{-\delta_{aa'}\rho^{\eta\xi}_{q+q'a}} \\
\left[\rho^{\eta\xi}_{qa}, \rho^{\xi\eta}_{q'a'}\right] &= -\frac{\delta_{aa'}}{2}\left(\rho^{\xi\xi}_{q+q'a} + \frac{1}{2}\mathrm{sgn}(a)\delta_{q+q',0}\frac{qL}{2\pi}\right),
\end{aligned}
$$

These terms have zero expectation value on GS, while necessary to close the algebra!

- (2)Response: How to couple the Hamiltonian to gauge field
- (3)Generalization to other model: MMHK, 1D Hubbard, higher dimension?

Thank You!