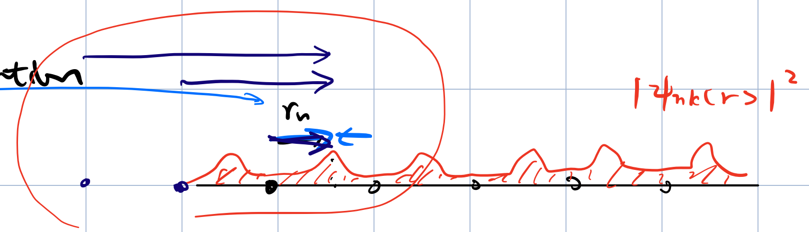
Topological Insulator & Bulk-boundary Correspondence

Review of last Lecture

- Bloch Band Theory $\begin{cases} \text{Plane Wave Expansion} \\ \text{Tight binding Approximation} \end{cases}$

- Modern Theory of Polarization

$$A_n = \langle u_n | i\nabla_k | u_n \rangle$$



$$\hat{r} \longleftrightarrow \hat{k} \sim i\nabla_r$$

$$r_n = \langle \psi_{nk'} | \hat{r} | \psi_{nk} \rangle = \langle u_{nk'} | e^{-ik'\hat{r}}\, \hat{r}\, e^{ik\hat{r}} | u_{nk} \rangle$$

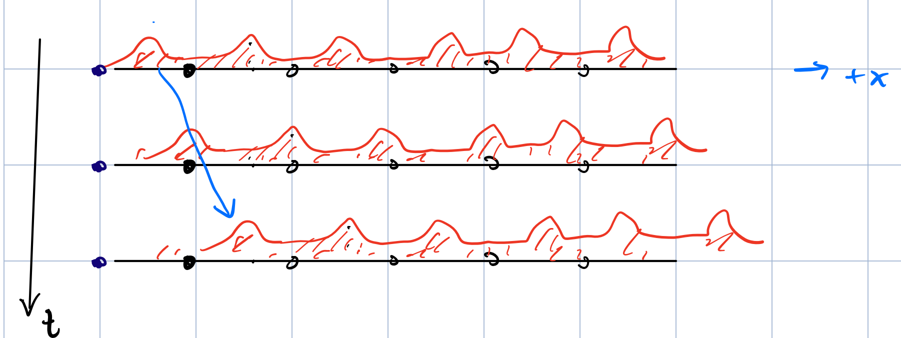
$$= \langle u_{nk'} | e^{-ik'r} \left(-i\nabla_k (e^{ik\cdot r})\right) | u_{nk} \rangle$$

$$= \langle \psi_{nk'} | -i\nabla_k | \psi_{nk} \rangle + i \langle u_{nk'} | e^{-ik'r}\, e^{ik\hat{r}} | \nabla_k u_{nk} \rangle$$

$$= i\nabla_k \langle \psi_{nk'} | \psi_{nk} \rangle$$

$$= i\nabla_k\, \delta(k-k') \qquad + \quad A_n(k)$$

Gauge-dependence $\quad \psi_{nk} = e^{ikr} u_{nk} \longrightarrow \psi'_{nk} = e^{ik(r-r_0)} u_{nk}$



Current = change of P

Quantized Hall Conductivity $\qquad j_y = \sigma_H E_x$

$$j_y = \frac{dP^y}{dt} = \frac{d}{dt}\left(e \sum_{occ.} \int_k \langle \psi_{nk} | \hat{y} | \psi_{nk} \rangle \right)$$

$$= \frac{d}{dt}\left(\sum_{occ.} \int_k A^y_{nk} \right) \qquad k \rightarrow k + \frac{eE}{\hbar} t$$

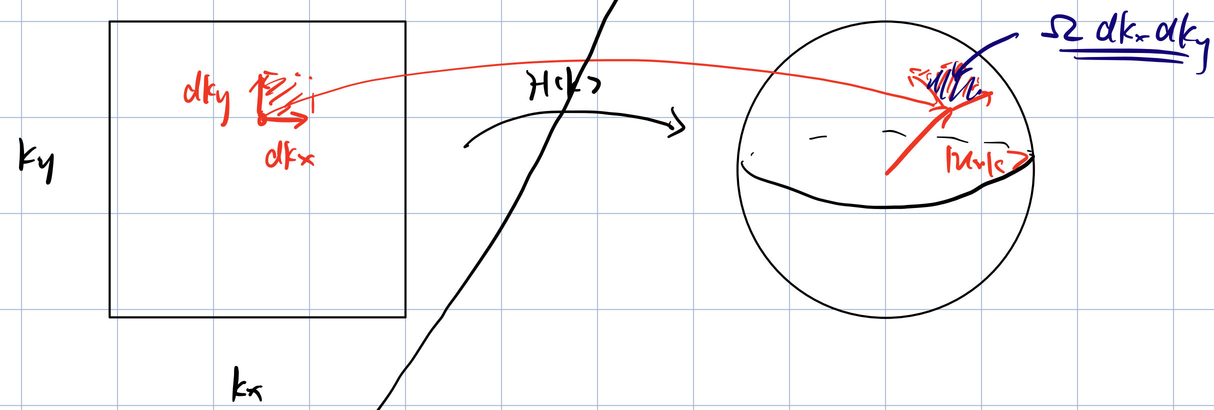
$$\frac{d}{dt}|\psi_{nk}\rangle = |\partial_{k_x}\psi_{nk}\rangle \frac{dk_x}{dt} = \frac{eE_x}{\hbar}$$

$$= \sum_{occ.}\int_k (\partial_{k_x}A^y - \partial_{k_y}A^x)\frac{eE_x}{\hbar}$$

$$= \left(\sum_{occ.}\int_k \Omega^{xy}\right)\frac{e^2E_x}{\hbar}$$

$$\langle \partial_{k_x} u_{nk}|\partial_{k_y} u_{nk}\rangle - (x \leftrightarrow y)$$

Quantized $\quad \sigma_H = \frac{e^2}{\hbar}\int d^2k\, \Omega = \int_{\text{Bloch}} dS = C \in \mathbb{Z} \leftarrow$ Topology



Topology. this doesn't change under continuous / smooth perturbation

$f(E^0)$ $\quad$ $f(E^1)$

A. Kubo Formalism

$$\sigma_{xy} = \int d^2k \frac{\langle u_n|\frac{\partial H}{\partial k_x}|u_m\rangle\langle u_m|\frac{\partial H}{\partial k_y}|u_n\rangle}{(\epsilon_n-\epsilon_m)^2}$$

$$= \int d^2k\, \Omega^{xy}$$

B. Boltzmann Transport

$$\partial_t f + eE\cdot\nabla_k f = \frac{f-f_0}{\tau}$$

$$j = \int f \cdot v$$

$$v_n = \frac{\partial\epsilon_n}{\partial k} + eE\times\Omega \quad (E^1)$$

(E^0)

$$\int f\, eE\times\Omega$$

$$j(E^1) = f^{(1)}v^{(0)} + f^{(0)}v^{(1)}$$

$$f^{(1)} = eE\tau\nabla_k f^{(0)}$$

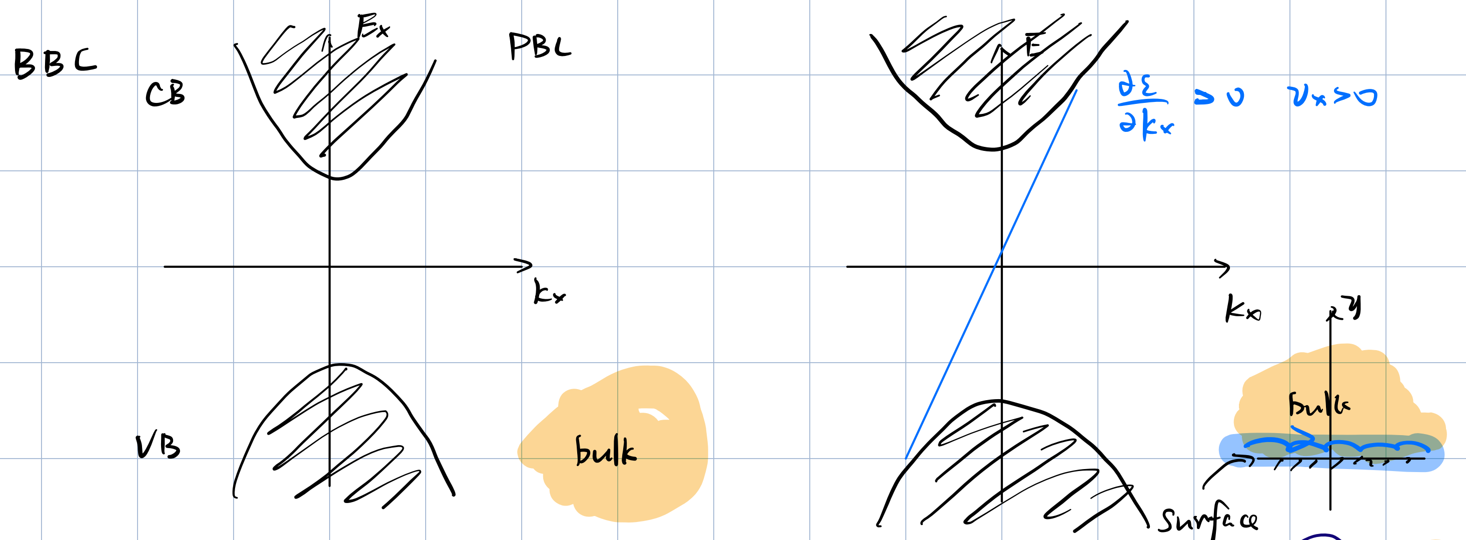
$$v^{(0)} = \frac{\partial\varepsilon}{\partial k}$$

$$j_{\text{Drude}} = eE\tau\int_k(\nabla_k f)(\nabla_k\varepsilon)$$

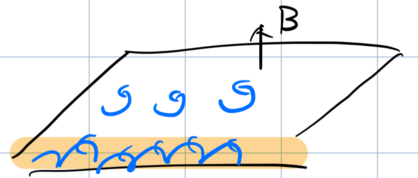
$$= eE\tau \int_k f\,(\partial_k v_k \varepsilon) \qquad \frac{\hbar^2 k^2}{2m}$$

$$= \frac{eE}{m^*}\tau \left[\int f\right]$$

$$= \boxed{\frac{neE}{m^*}\tau}$$



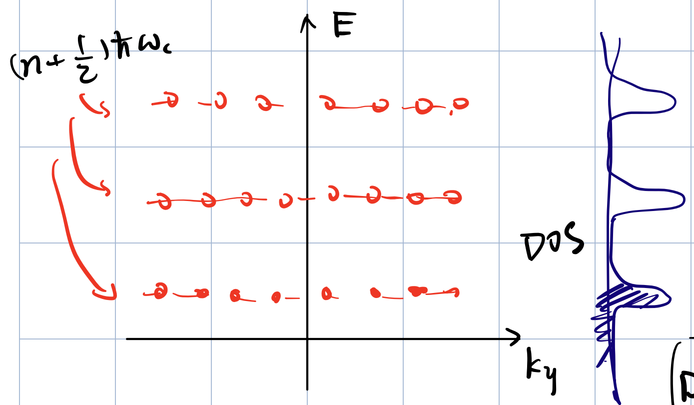
$$(x - \frac{p_y}{B})^2 \qquad \frac{(p_y - Bx)^2 + p_x^2}{2m}$$

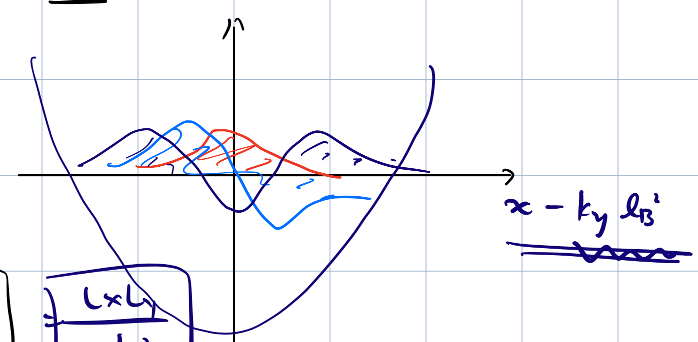


★ Bulk $\quad H = H(p \to p + eA)$

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{A} = -Bx\hat{y}$$

$$\psi_{n,k} = e^{ik_y y}\, H_n(x - k_y l_B^2)$$

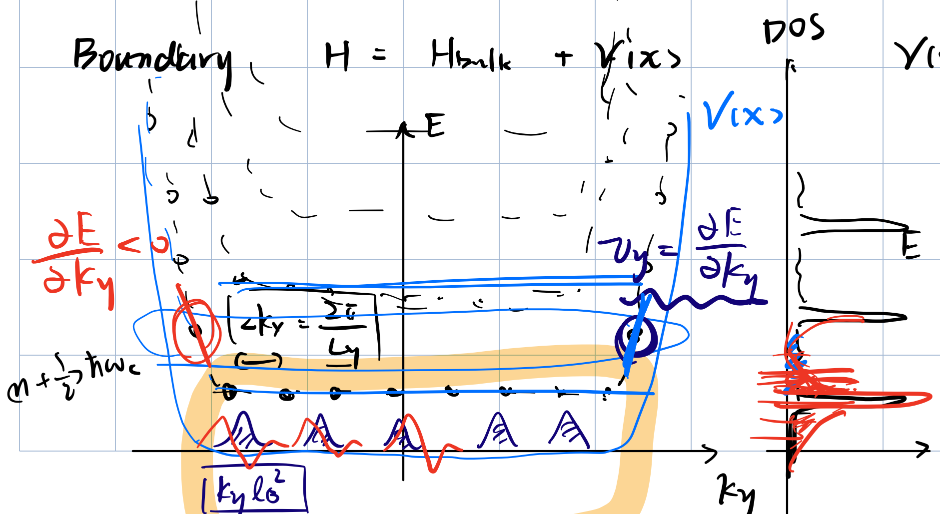




$$\boxed{D = \frac{S}{\pi l_B^2}} \quad = \boxed{\frac{L_x L_y}{\pi l_B^2}}$$

Boundary $\quad H = H_{bulk} + V(x)$

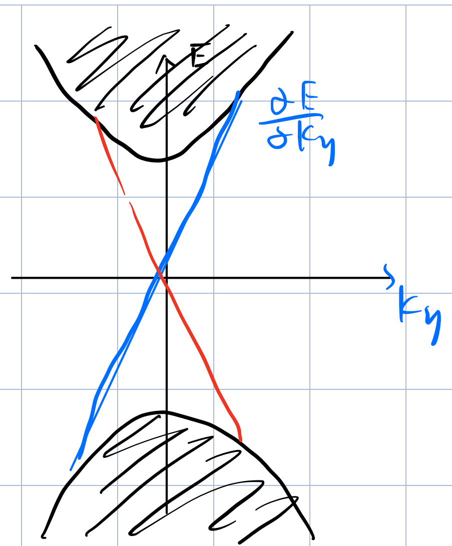
$$V(x) = \begin{cases} 0 & \text{bulk} \\ \infty & \text{outside} \end{cases}$$

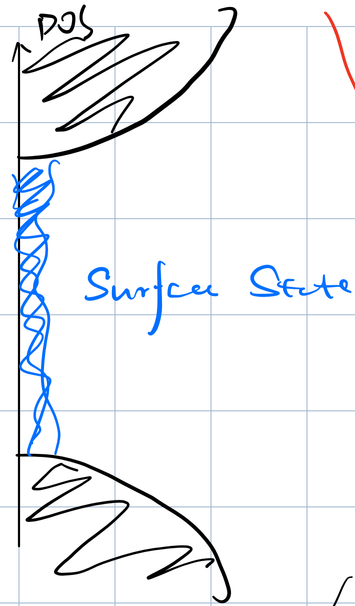


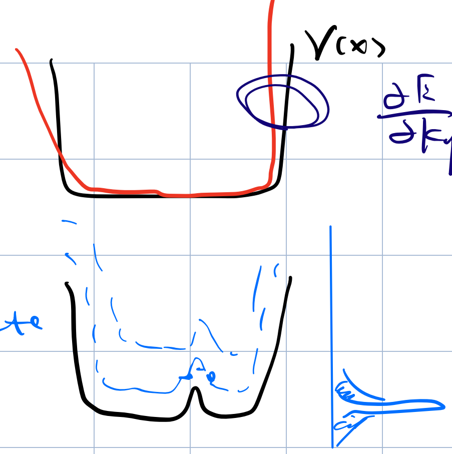
$$E = \langle\psi|H|\psi\rangle \qquad \int \psi(x)\, V(x)\, \psi(x) \qquad \delta(x - k_y l_B^2)$$

$$= \langle\psi|H_{bulk}|\psi\rangle + \langle\psi|V(x)|\psi\rangle$$

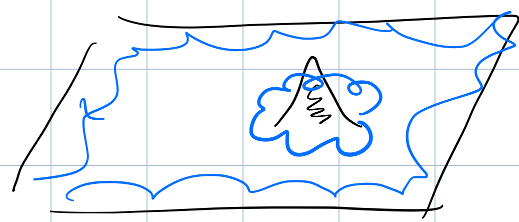
$$= (n+\tfrac{1}{2})\hbar\omega_c + V(k_y l_B^2)$$

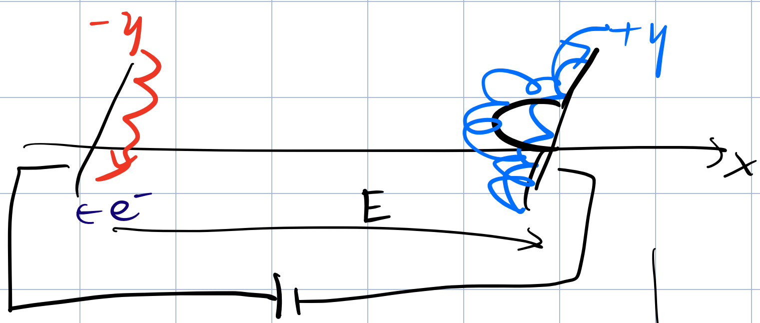




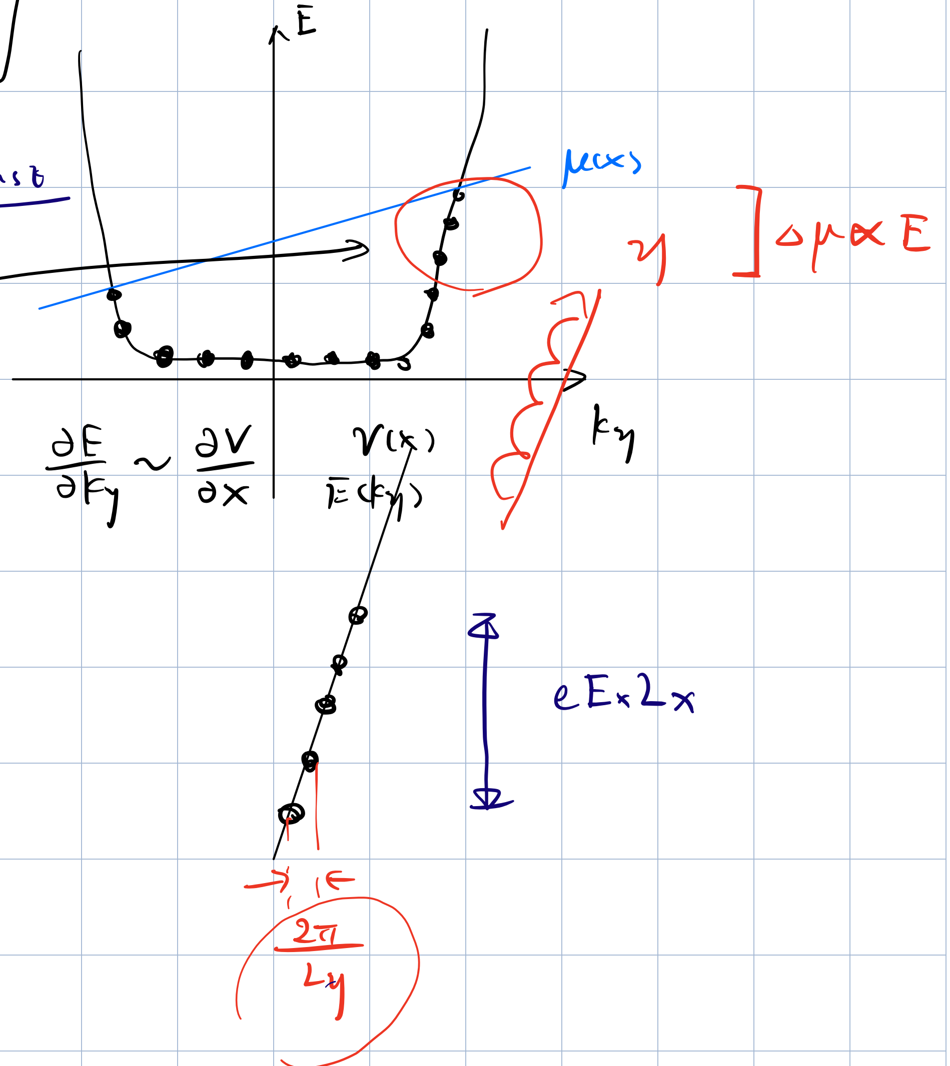


$$\frac{\partial E}{\partial k_y} \sim \frac{\partial V}{\partial x}$$





$$\mu(x) + eEx = \text{Const}$$



$$\frac{\partial E}{\partial k_y} \sim \frac{\partial V}{\partial x}$$

$$I_y = \Delta N e v_y = \cancel{\frac{\partial E}{\partial k_y}}$$

$$\Delta N = \frac{eE_xL_x}{\cancel{\frac{\partial E}{\partial k_y}} \cdot \frac{2\pi}{L_y}}$$

$$= \frac{eE_xL_xL_y}{2\pi} e$$

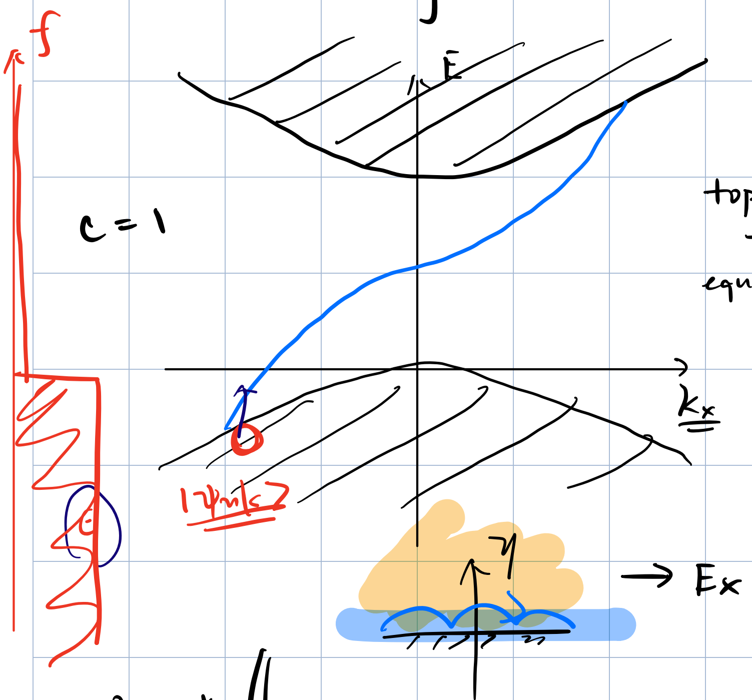
$$j_y = \frac{I_y}{L_x}$$

$$\sigma_{xy} = \frac{j_y}{E_x} = \frac{e^2}{h}$$

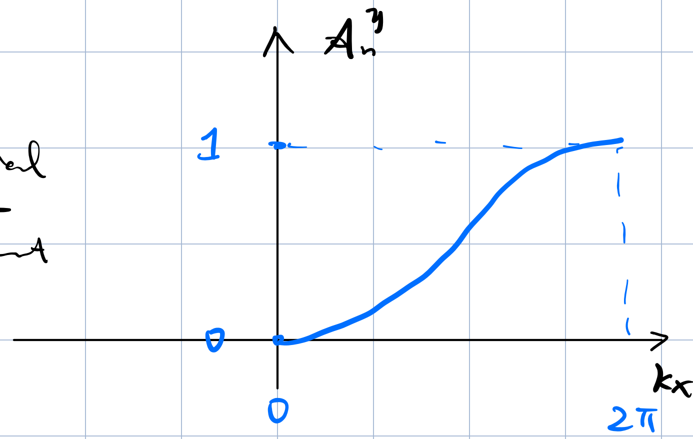
boundary ⟷ bulk

Boundary Flow $\Longleftrightarrow$ Pump of Polarization

$c = 1$



topological $=$ equivalent

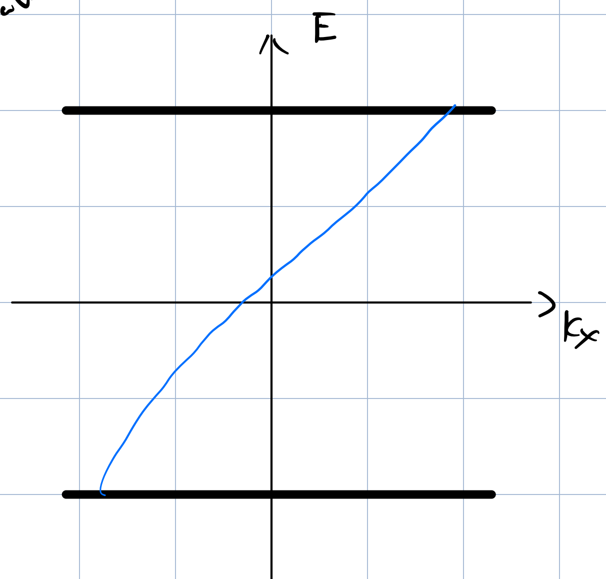


$$\hat{H}(k) = \sum E_n(k)\, |\psi_{nk}\rangle\langle\psi_{nk}|$$

$$\downarrow$$

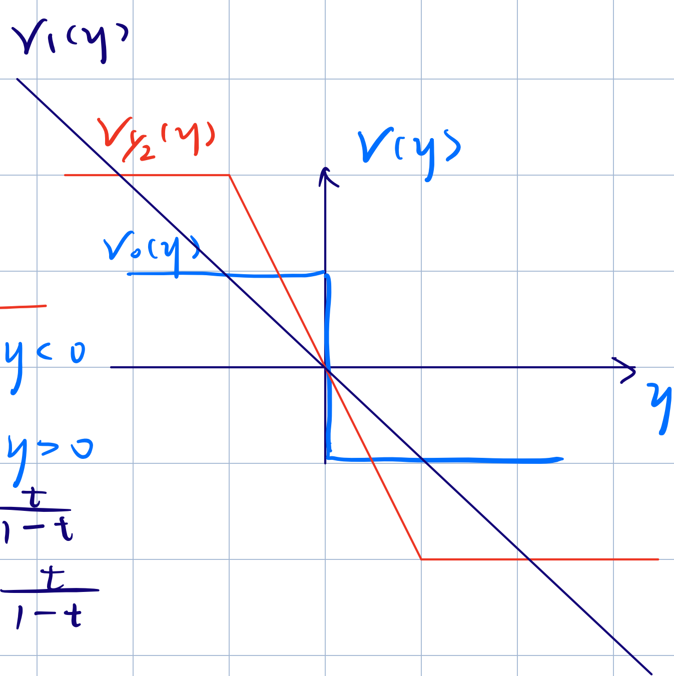
$$\sum E_{\text{const}}\, |\psi_{nk}\rangle\langle\psi_{nk}|$$

topo equivalent $\parallel$ $\qquad$ $/\!/$?



$$\mathcal{H}^{\text{bulk}}_{\text{flat}} = 1 - 2\hat{P}_{\text{occ.}}$$

$$\hat{P}_{\text{occ.}} = \sum_{\text{occ.}} |\psi_n\rangle\langle\psi_n|$$



$t=0$

$$\mathcal{H}^{\text{boundary}}_{\text{flat}} = \hat{P}\, V_0(\hat{y})\, \hat{P} + 1 - \hat{P}$$

$$V_0(\hat{y}) = \begin{cases} 1 & \text{outside } y<0 \\ -1 & \text{bulk } y>0 \end{cases}$$

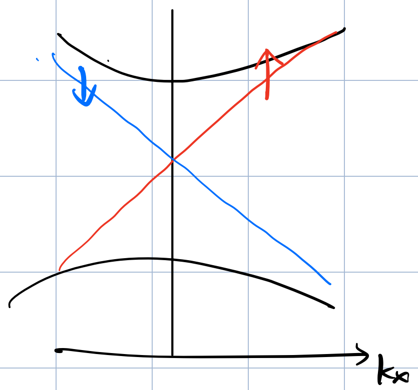
$$V_t(y) = \begin{cases} -\dfrac{y}{t} & |y| < \dfrac{t}{1-t} \\ -\dfrac{\operatorname{sgn}(y)}{1-t} & |y| \geq \dfrac{t}{1-t} \end{cases}$$

$t = 1/2$

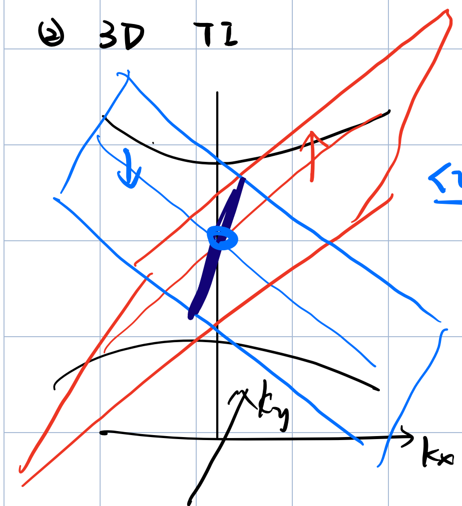
$$H^{t=1} = -\hat{P}\,\hat{y}\,\hat{P} + 1 - \hat{P} \xrightarrow{\text{spectrum}} \langle\psi_n|\hat{y}|\psi_n\rangle$$

$$\langle\psi_n|\Big(|\tau\psi_n\rangle\langle\psi_n|\tau|\psi_n\rangle\langle\psi_n| + \mathbb{1} - |\psi_n\rangle\langle\psi_n|\Big)\tau\psi_n\rangle$$

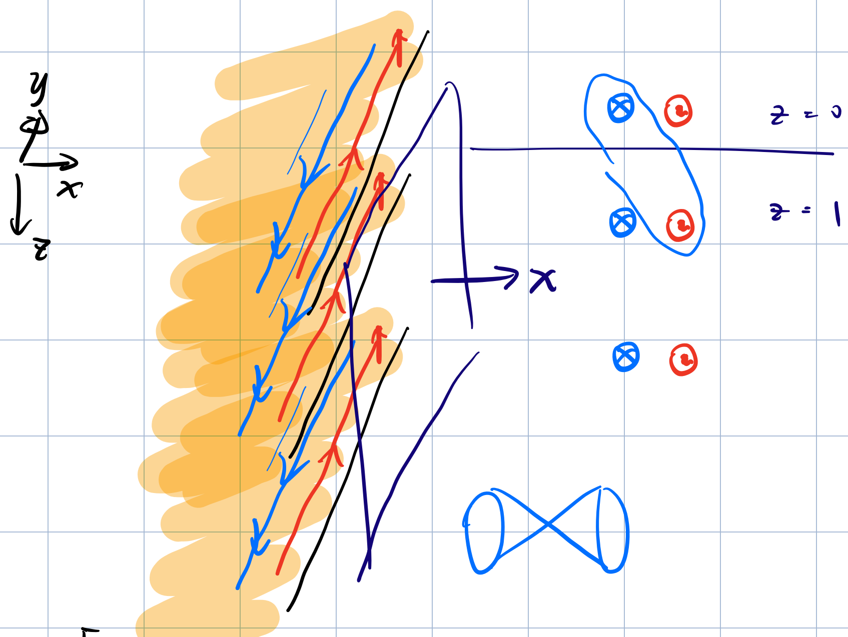
① 2D TI

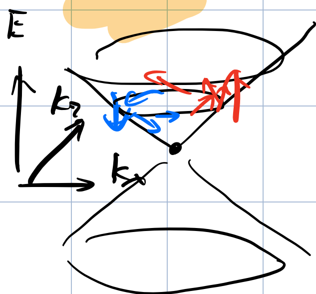


② 3D TI



$\langle\psi|\mathcal{T}\psi\rangle = 0$





③ Weyl Semimetal

④ HOTI